INVERSE REASONING WITH QUANTITATIVE unknowns

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IDR^2eAM Project

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IDR² eAM Project

Investigating Differentiated Instruction and Relationships between Rational Number Knowledge and Algebraic Reasoning in Middle School

• **Purposes: To study...**
  • How to differentiate instruction for cognitively diverse middle school students
  • How students’ rational number knowledge and algebraic reasoning are related

• **Phase I (2 yrs):** Three design experiments after school
  • 6-9 seventh & eighth grade students selected for cognitive diversity
  • 18 episodes each, 22 students total
Purpose of Talk

• How was reflective abstraction involved in constructing and stabilizing reciprocal reasoning with quantitative unknowns?
• What about students who did not construct reciprocal reasoning?

• **Reciprocal reasoning**: seeing, justifying, and using the idea that if \( \frac{3}{5} \) of height B is height A, then B must be \( \frac{5}{3} \) of A (more soon)
Mathematical Learning

- I view learning in the context of making accommodations in schemes in on-going interaction in one’s experiential world.
- **Schemes:** goal-directed ways of operating that include a perceived situation, activity, and perceived result.
- **Accommodations:** reorganizations of and modifications in schemes

Steffe & Olive (2010, p. 23)
The Iterative Fraction Scheme

- Fractions are *multiples* of unit fractions
- Improper fractions are also whole numbers with additional fractional parts
- Students can think about and operate with fractions beyond 1 without conflations.

Unit Bar

<table>
<thead>
<tr>
<th>five-fifths, a unit of five units</th>
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</thead>
<tbody>
<tr>
<td>seven-fifths, a unit of seven units</td>
</tr>
<tr>
<td>three-fifths, a unit of three units</td>
</tr>
<tr>
<td>one-fifth</td>
</tr>
<tr>
<td>one-fifth</td>
</tr>
<tr>
<td>five-fifths, a unit of five units</td>
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<tr>
<td>one-fifth</td>
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</tbody>
</table>
Reflective abstraction (Piaget, 1980)

- **Reflecting abstraction**: reorganization of mental operations and projection of them to new level
  - Motor behind construction of schemes (Thompson, 1994)
  - Motor behind some accommodations
  - Motor behind interiorization of results of schemes = formation of concepts

- **Reflected abstraction**: deliberate thematization of mental operations
  - “Looking back” on one’s ways of thinking to discern patterns and structure
Phase I students: 9 out of 22 students with IFS

<table>
<thead>
<tr>
<th></th>
<th>IFS, initially</th>
<th>RR</th>
<th>IR</th>
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<tbody>
<tr>
<td>Experiment 1</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Experiment 2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Experiment 3</td>
<td>3</td>
<td>2</td>
<td>1</td>
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</tbody>
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**Conjecture:** These students will construct reciprocal reasoning with quantitative unknowns.

**Finding:**
- 7 of the 9 students constructed reciprocal reasoning;
- 2 constructed inverse reasoning with fractional relationships between quantitative unknowns.
Reciprocal Reasoning with Quantitative Unknowns

**Problem (summary):** The unknown height of the sunflower is $3/5$ the unknown height of the fern.

- Draw a picture.
- Write and explain equations.
- All wrote an equation with $3/5$.
- Two students knew that $5/3$ would be the other relationship to use; none could justify it originally.
- Seven students experienced an insight that each $1/5$ of the fern ht was $1/3$ of the sunflower ht.

Let $x =$ fern ht, $y =$ sunflower ht

$y = 3/5x \text{ AND } x = 5/3y$
Reciprocal Reasoning Scheme

Develop equations for 2 Unknown Problems

Make fractions of unknowns

Use IFS recursively

Create 2 equations with reciprocal relationship

• 7 made initial construction
• 4 stabilized their schemes
Account in terms of Reflecting and Reflected Abstraction

- **Initial construction of reciprocal reasoning scheme:** accommodation in iterative fraction scheme – province of REFLECTING ABSTRACTION

- **Subsequent stabilization of reciprocal reasoning scheme:** deliberate thematization – province of REFLECTED ABSTRACTION

- But what happened with 2 of the 9 students, Amanda (e2) and Katrina (e3)?
Inverse Reasoning with Quantitative Unknowns: Amanda (e2) and Katrina (e3)

<table>
<thead>
<tr>
<th></th>
<th>Equations for Fern Sunflower Heights Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amanda</td>
<td>$f = \text{fern ht}; s = \text{sunflower ht}$</td>
</tr>
<tr>
<td></td>
<td>$f \div 5 \cdot 3 = s, s \div 3 \cdot 5 = f \leftarrow$ then rejected this</td>
</tr>
<tr>
<td></td>
<td>Sequence: $s + 2/5 = f, s + f2/5 = f, s + 2/3 = f, \text{ and } s + 2/3s = f$</td>
</tr>
<tr>
<td>Katrina</td>
<td>$f \div 5 \cdot 3 = s, s \div 3 \cdot 5 = f$</td>
</tr>
<tr>
<td>Katrina, ht A is 2/7 of ht B problem:</td>
<td>$A + A + A + 1/2A = B$</td>
</tr>
</tbody>
</table>
Katrina’s Follow-up Interview

• **Problem:** Steven and Lia are each growing a sunflower plant. The height of each of their plants is unknown, and the height of Lia’s plant measured in inches is \( \frac{3}{7} \) the height of Steven’s plant measured in inches. [Draw a picture, write equations.]

• Katrina’s initial equations:
  • \( S ÷ 7 \times 3 = L \)
  • “an opposite of this,” \( L ÷ 3 \times 7 = S \)
  • With prompting to use a fraction:
    • \( \frac{3}{7}S = L \)
Katrina’s equations so far:

\[ S \div 7 \times 3 = L \]
\[ L \div 3 \times 7 = S \]
\[ 3/7S = L \]
Inverse v. Reciprocal Reasoning

- **Reciprocal (stable):** package of two relationships
- **Reciprocal (initial construction):** package of relationships that needs to be recreated
- **Inverse:** process represented to produce each height from the other, usually with whole number multiplication and division
  - From Inhelder and Piaget’s inversion aspect of reversibility (1958)
Oops!

• **Conjecture:** Need to support construction/stabilization of iterative fraction scheme before working on reciprocal reasoning

• Too much attention to designing to support reflected abstraction v. designing to support reflecting abstraction

• Is it a short-term learning goal for students like Amanda and Katrina to stabilize their iterative fraction schemes?
THANK YOU!

• To co-author on the reciprocal reasoning paper, Serife Sevis
• And BIG thanks to others on the IDR²eAM project team: Fetiye Aydeniz, Rebecca Borowski, Mark Creager, Ayfer Eker, Sharon Hoffman, Robin Jones, Rob Matyska, Pai Suksak
• What IDR²eAM stands for: Investigating Differentiated Instruction and Relationships between Rational Number Knowledge and Algebraic Reasoning in Middle School
• http://www.indiana.edu/~idream/
Revised Learning Trajectory for RR with QU

<table>
<thead>
<tr>
<th>Significant events</th>
<th>Description of the reasoning</th>
<th>Learning Processes</th>
<th>Instructional Supports (Examples)</th>
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</table>
| Constructing or solidifying an iterative fraction scheme and reversible iterative fraction scheme | Students view any fraction as a multiple of a unit fraction, as well as a number of units of 1 and a proper fraction. | Initial construction: An accommodation in students’ fraction schemes in which students see the result of their scheme as a multiplicative relationship, rather than as parts out of a whole. Stabilization: Reflected abstraction where students engage in repeated experience and retroactive thematization to examine and use improper fractions as usable numbers. | • Asking students to draw $\frac{7}{5}$ of a bar, or to draw the whole bar given $\frac{7}{5}$ of it  
  • Asking students to iterate unit fractions in order to create improper fractions  
  • Asking students to, e.g., use ninths to draw a bar a little bit longer than an $\frac{8}{8}$-bar |