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Understanding How to Differentiate Instruction for Middle School Students

Today's middle school mathematics classrooms are marked by increasing cognitive diversity. Traditional responses to cognitive diversity are tracked classes that contribute to opportunity gaps (Flores, 2007) and can result in achievement gaps. Differentiating instruction (DI) is a pedagogical approach to effectively manage classroom diversity, in which teachers proactively plan for adapting curricula, teaching methods, resources, student activities, and products of learning to address individual students' needs in an effort to maximize learning for all students (Heacox, 2002; Tomlinson, 2005). So, DI involves systematic forethought rather than only reactive adaptation.

Principles of DI include that it is rooted in on-going formative assessment, that it positions teacher and students as learners together, that it emphasizes engaging all students in critical and creative thinking, and that it requires teachers to clarify big ideas for instruction in order to make effective adaptations of curricula and teaching methods (Santamaria, 2009; Tomlinson, 2005). These principles are consistent with reform efforts in mathematics education (National Council of Teachers of Mathematics [NCTM], 2000, 2014), and overall DI is a possible response to NCTM's (2000) principle of equity. However, as of yet differentiated mathematics instruction has not been systematically investigated, especially at the secondary level. Indeed, secondary mathematics classrooms are places where differentiation is least likely to occur (Gamoran & Weinstein, 1998).

The purpose of this paper is to report on this research question: **How did pedagogical activities facilitate and impede differentiating mathematics instruction for middle school students in an after school design experiment?** The report comes from a 5-year project to study DI and relationships between students' rational number knowledge and algebraic reasoning. In the first two years of the project we conducted three 18-episode design experiments, each with nine cognitively diverse 7th and 8th grade students.[1] The data for this paper come from the first experiment, highlighting four episodes in which students worked on representing multiplicative relationships between quantitative unknowns.

Rationale for and Research on DI

In this section we elaborate our rationale for DI and our working definition of DI in the study. Then we review literature on and related to DI.

Rationale for DI

Tracking practices have been shown to have negative effects on students (Oakes, 2005). Low-achieving students are often placed in mathematics classes with undue emphasis on rote learning and computation, little or no engagement with higher demand mathematical tasks, and few possibilities for movement to classes with higher achieving peers (e.g., Boaler, Wiliam, & Brown, 2000; National Research Council [NRC], 2001; Stiff, Johnson, & Akos, 2011). Some students in advanced tracked classes experience disaffection and low self-concepts (Boaler et al., 2000; Mulkey, Catsambis, Steelman, & Crain, 2005). Furthermore, recommendations for class placements have been found to contain bias against promoting minority students to advanced classes (Flores, 2007; Rubin, 2006; Stiff, et al., 2011). Students in lower classes tend to fall further and further behind their higher achieving peers, a phenomenon referred to as a significant opportunity gap (Flores, 2007; NCTM, 2012).

In contrast, mathematics instruction in heterogeneous groups may help low achieving students learn more (e.g., Boaler, 2006; Boaler & Staples, 2008; Burris, Huebert, & Levin, 2006; Burris, Wiley, Welner, & Murphy, 2008; Linchevski & Kutscher, 1998). In some studies researchers have found that high achieving

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[1] All experiments began with nine students, and all nine completed the first experiment. Seven of the nine students completed the second experiment, and six of the nine students completed the third experiment.

students' achievement suffers when mathematics classes are conducted heterogeneously (e.g., Loveless, 2009; Nomi & Allensworth, 2013). However, in other studies, high achieving high school students benefitted from detracked mathematics classes, where benefits included higher achievement on curriculum-based tests compared to students at tracked schools, increased enjoyment of mathematics, appreciation of students from a variety of backgrounds (Boaler & Staples, 2008), as well as significant increases in earning advanced diplomas (Burris, et al., 2008).

Yet simply placing cognitively diverse students in the same classroom and teaching from a curriculum with high demand tasks does not address the significant issues that students may experience due to their different learning needs (Gamoran & Hannigan, 2000; Lawrence-Brown, 2004; NRC, 2001). So, teaching cognitively diverse students in one classroom must be done with great skill and care in order to fulfill a promise of being more humane than tracking (Daniel, 2007; Mevarech & Kramarski, 1997; Rubin, 2008; Staples, 2008), and to have the potential to close opportunity gaps.

Definition of DI

Tomlinson's (2005; Tomlinson et al., 2003) framework for DI features differentiating three facets of instruction—content, process, and products—based on three characteristics of students: cognitive readiness, interests, and learning profile, which is defined as the ways students learn best, such as learning style or preferences (Gardner, 1993). In our project we are focused on differentiating for cognitive diversity. So, following Tomlinson (2005) our working definition of DI is *proactively tailoring instruction to students' mathematical thinking while aiming to develop a cohesive classroom community*.

We view "tailoring instruction" to include posing problems that are in harmony with students' thinking while also posing challenges at the edge of students' thinking (cf. Hackenberg, 2010a). Tailoring instruction relies on on-going formative assessment as a central tool in teaching, which is consistent with definitions of DI by a range of researchers (e.g., Heacox, 2002; Roy, Guay, & Valois, 2013; Smit & Humpert, 2012; Tomlinson, 2005). We also view tailoring instruction to include interactions with individual students and groups of students during a class session in which the teacher questions and prompts in a responsive way (cf. Jacobs & Empson, 2015; Smit & Humpert, 2012).

Although not all researchers include the component of "a cohesive classroom community" in their definitions (e.g., Roy, et al., 2013; Smit & Humpert, 2012; Tomlinson et al., 2003), we believe this component is important because we want to avoid creating "tracks" of students within a single classroom. That is, one way to tailor instruction would be to run four different classes in each corner of the classroom, where the same groups of students always stay in the same corner. Or to move to a different extreme, another way to tailor instruction would be to develop an individualized education plan for each student, where everyone works individually all the time. We don't view either of these extremes as what we mean by DI because of the lack of interaction and communication between groups or individuals.

Instead, in our view of DI students move in and out of different grouping arrangements (Tomlinson, 2005), and the teacher seeks ways to use thinking from individuals and small groups to shape whole classroom interactions. We view a "cohesive classroom community" to be a collection of people who feel connected to each other as mathematical thinkers and a sense of belonging (Baumeister & Leary, 1995; Davis, 2003; Goos, 2004). Following Cobb (2000), we view individual students' reasoning and the microculture of the classroom community to be reflexively related.

Research on DI

Since DI is a relatively new sub-field in education, evidence of its effectiveness currently rests on many testimonials (e.g., Lawrence-Brown, 2004; Laud, 2011; Wormell, 2006) and some positive findings (e.g., Baumgartner, Lipowski, & Rush, 2003; Gearhart & Saxe, 2014; Reis, McCoach, Little, Mueller, & Kaniskan, 2011; Santamaria, 2009; Tieso, 2005). For example, in Gearhart and Saxe's Learning

Mathematics through Representations (LMR) project, the authors designed a sequence of 19 lessons for 4th and 5th grade students focused on developing students' understanding of number lines as a core representation for learning fractions and integers. Each lesson featured key aspects of differentiation, such as problems that were accessible to a range of students; a variety of individual, small group, and whole classroom instruction; and on-going formative assessment. In an efficacy study with a well-matched comparison group, the authors found that students in LMR classrooms showed significantly greater learning gains than students in comparison classrooms (Saxe, Diakow, & Gearhart, 2013), although it's not clear whether the approach to number lines or the DI or a combination of the two produced the effect.

Similarly, Tieso (2006) conducted a quasi-experimental study with 31 4th and 5th grade teachers and their 645 students to examine the effects of ability grouping (whole, between-class, and within-class) and curricular practices (traditional, revised, and differentiated) on students' achievement on a data representation and analysis unit. Tieso found that differentiating curriculum and instruction by readiness helped keep high-ability students challenged in heterogeneous classrooms and yielded significantly higher scores for regularly-achieving and high-achieving students on a curriculum-based test; scores of low-achieving students increased but not significantly.

Outside of mathematics classrooms, Reis and colleagues (2011) conducted an experimental study on reading achievement in five elementary schools located across the United States. They found that replacing 5 hours per week of reading instruction in grades 2-5 with a school-wide enrichment model-reading (SEM-R) program and activities designed to differentiate instruction in reading resulted in no difference in student outcomes at three schools, improved reading fluency at one school, and improved reading fluency and comprehension at another high-poverty, urban school. In addition, qualitative data from teacher participants revealed that a critical impact of differentiated activities was the creation of an engaging and joyful learning environment for reading. Baumgartner and colleagues (2003) found similar positive results for low-achieving elementary and middle school students who experienced a differentiated reading program, including an improvement in students' attitudes toward reading.

Studies across domains underscore that even when positive influence on student achievement is not uniformly found (e.g., Reis et al., 2011; Tieso, 2006), researchers often find some positive influences on other student outcomes such as enjoyment or attitudes. For example, Roy and colleagues (Roy, Guay, & Valois, 2015) found that teacher self-reports of the use of DI strategies was associated with eliminating the effects of factors that diminish academic self-concept for struggling students in French class. As another example, in case studies of two different elementary schools with high populations of culturally and linguistically diverse learners Santamaria (2009) found that a combined approach of DI and culturally responsive teaching (Gay, 2000; Ladson-Billings, 1994, 2001) was associated with improved student achievement over a 5-year period. However, there were many other outcomes as well, such as greater respect for students with learning disabilities and more positive cross-cultural student interactions.

Despite the promise of DI, implementation of it is challenging (Brighton & Hertberg, 2004; Gamoran & Weinsten, 1998; Tomlinson, 1995; Tomlinson et al., 1997). To begin with, teachers have different understandings of the nature of DI (Brighton & Hertberg; Tomlinson, 1995), which are formed in part by differences in their beliefs about teaching and learning, and which impact implementation. Furthermore, teachers are understandably concerned about several issues, such as management of the classroom environment, management of assessment information (Simpson, 1997), fairness of students doing different work and receiving different support (Hockett, 2010), and increased workload. Thus teachers, particularly at the secondary level, may indicate support for differentiated instruction while making few changes to incorporate differentiated instruction strategies into their practice (Simpson, 1997; Smit & Humpert, 2012; Tomlinson et al., 2003). Instead, when teachers do differentiate for struggling or advanced students they often alter the amount of work or adjust grading standards. This kind of differentiation does not usually serve students as well as a more proactive and comprehensive approach.

Research Related to DI within Mathematics Education

To our knowledge, the LMR study by Gearhart and Saxe (2014; Saxe et al., 2013) represents the only study on DI within Mathematics Education, and no research on DI has been conducted at the secondary level in the field. However, there are several closely related research areas, such as noticing (e.g., Sherin, Jacobs, & Philipp, 2011), responsive teaching, (e.g., Jacobs & Empson, 2015), and efforts to understand how to build on student thinking in instruction (e.g., Leatham, Peterson, Stockero, & Van Zoest, 2015).

Van Es and Sherin (2002) define noticing as "(a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions" (p. 573). Noticing is relevant to DI because all three components can be useful in tailoring instruction to student thinking. Research on noticing at the secondary level has mostly focused on how teachers' noticing skills may change over time with professional development (e.g., Goldsmith & Seago, 2011; van Es & Sherin, 2002, 2008). For example, van Es and Sherin (2008) have found that teachers may shift from an evaluative to an interpretive stance, while Goldsmith and Seago have found the nature of teachers' interpretations changed from lacking evidence to being rooted in evidence.

The noticing literature is a context for some recent studies of responsive teaching (Dyer & Sherin, 2015; Jacobs & Empson, 2015; Lineback, 2015; Pierson, 2008). Jacobs and Empson define responsive teaching as "a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking, instead of being determined in advance" (p. 1 of pdf). Dyer and Sherin (2015) studied two high school mathematics teachers who practiced responsive teaching and found they engaged in three kinds of instructional reasoning while teaching: making connections between multiple moments in students' thinking, pondering the relationship between student thinking and the structure of the task, and engaging in tests of their interpretations of student thinking. The authors noted that engaging in tests is a skill required in clinical interviews, supporting a link between responsive teaching skills developed in research-based environments and those that are useful in whole classrooms (cf. Jacobs and Empson).

We elaborate on this idea, noting that all three kinds of instructional reasoning Dyer and Sherin (2015) identified are embedded in design experiment and teaching experiment methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000). First, in conducting design experiments, a teacher regularly compares multiple moments in students' thinking in order to develop working models of students' thinking (Steffe & Thompson), consisting of a teacher's ways of thinking that she attributes to students as she gets to know them. These working models are a critical guide for on-going interaction with students in an experiment, and they are the basis for building more analytic models retrospectively. Second, since task design is a key goal of design experiments, the teacher and research team regularly consider task structures in relation to student thinking; this type of instructional reasoning is both planned and arises in the moment during episodes. Third, making conjectures about student thinking and engaging in tests of those conjectures is another built-in feature of design experiment and teaching experiment methodology that informs working models and interaction.

In our view, responsive teaching is one part of differentiating instruction because being responsive in the moment is one way of tailoring instruction to students. It may also have added benefits of helping to build classroom community because of the expectations and norms that develop around what mathematics class is about, and what the roles of students and teachers are. That is, if teachers are regularly responsive in their interactions with students, students may trust that the teachers will appropriately challenge them, posing questions that are neither out of reach nor too simple. Thus responsive teaching can be seen as related to mathematical caring relations, a kind of "cognitive care" that has been analyzed as important for mathematical learning (Hackenberg, 2005, 2010a). One issue that has recently received attention toward this end is how teachers can learn to use students' mathematical thinking productively in whole class instruction (Leatham, et al., 2015), an issue we return to later in the paper.

Theoretical Framework

In this section we present our view of mathematical thinking and learning, our use of students' multiplicative concepts as a key indicator of cognitive diversity, and the nature of hypothetical learning trajectories in the experiment.

Mathematical Thinking and Learning

For us, mathematical thinking consists of coordinations of mental actions, or operations (Piaget, 1970; von Glasersfeld, 1995). Operations are the components of schemes, goal-directed ways of operating that involve a situation as perceived and conceived by the learner, activity, and a result or outcome that the learner assesses in relation to her goals (von Glasersfeld). We view mathematical learning in the context of making reorganizations, or accommodations, in schemes in on-going interaction in one's experiential world. Social interaction can open possibilities for accommodations and make operations and schemes apparent via verbalizations, non-verbal interaction, drawn representations, or mathematical notation.

Students' Multiplicative Concepts

In the experiment cognitive diversity was based primarily on students' multiplicative concepts, which are the interiorized results of students' units coordinating schemes (Hackenberg & Tillema, 2009; Steffe, 1994).[2] Broadly speaking, students enter middle school operating with three different multiplicative concepts that have significant implications for how students develop their rational number knowledge (e.g., Hackenberg, 2010b; Norton & Wilkins, 2012; Steffe & Olive, 2010); research is beginning to uncover implications for students' algebraic reasoning (e.g., Hackenberg & Lee, 2015; Olive & Caglayan, 2008). Transitioning between these concepts requires a significant reorganization of thinking that can take two years (Steffe & Olive, 2010). Steffe (2007) estimates that at the start of 6th grade, 70% of students are operating with the second or third multiplicative concepts—i.e., are MC2 or MC3 students. We identified students operating with the first multiplicative concept in our selection process, but none of these students opted to participate in the design experiments. So we focus on MC2 and MC3 students here.

MC2 students. A units coordination involves two composite units, such as 4 and 5, and it entails distributing the units of one composite unit across the elements of another composite unit (Steffe, 1992, 1994)—for example, distributing 4 units of 1 across each of the units of the 5 to get a unit of 20 that may be structured in various ways by students. Some students view the result of five 4s as a composite unit in which the elemental units of 1 are *iterable*, which means that for these students, there is a multiplicative relationship between elemental units of 1 and composite units. So, for these students, 20 is a number that can be created by iterating 1 20 times. Students who view the results of a units coordination as a composite unit consisting of iterable units of 1 have interiorized units of units, or two levels of units.

These students can treat lengths as a unit containing some number of equal units prior to operating in a situation (Steffe & Olive, 2010). For example, they can imagine a 1-meter length partitioned into five equal parts without making the partitions. Furthermore, they can partition each of those parts into some number of parts in the process of solving the problem, so they make three levels of units in activity. For example, they might partition each of the 5 parts into 4 mini-parts to produce a 1-meter length containing 20 mini-parts. But in further operating the three levels of units that may be visible to a teacher or researcher are not salient for them, and they work with two levels of units. So, in further operating they conceive of the 1-meter length as a unit of 20 units.

[2] *Interiorization* refers to re-processing the result of a scheme so that students can anticipate it prior to activity.

MC3 students. Students operating with the third multiplicative concept can take three levels of units as given and flexibly switch between three-levels-of-units structures (Hackenberg, 2010b; Steffe & Olive, 2010). So, in the whole-number coordination of five 4s, MC3 students can take as given both the distribution of five 4s and the structure of the result, 20, as a unit of five units, each of which contains four units. In short, MC3 students can operate strategically on different organizations of the 4s as if they were units of 1 without losing track of the 4s as composite units.

Prior to operating, MC3 students can also treat a length as a unit containing some number of units, each of which contains some number of units. So, in the example of the 1-meter length above, MC3 students can operate as MC2 student do, but they maintain a view of the length as a unit of five units each containing 4 units, and they can switch to viewing the length as a unit of 4 units each containing 5 units. Being able to flexibly switch between such unit structures is critical for constructing many fraction schemes (e.g., Hackenberg & Tillema, 2009; Steffe & Olive, 2010).

Other features of cognitive diversity. We also selected for diversity in other schemes and operations we could attribute to students, such as their splitting operations (Hackenberg, Jones, Eker, & Creager, in revision; Norton, 2008; Steffe, 2002) and fraction schemes (Hackenberg, 2007; Steffe & Olive 2010).

Hypothetical Learning Trajectories

Simon (1995) coined the term *hypothetical learning trajectory* (HLT) to "refer to the teacher's prediction as to the path by which learning might proceed" (p. 131). In a footnote, he explained that he chose to use the phrase *hypothetical learning trajectory* "to emphasize aspects of teacher thinking that are grounded in a constructivist perspective and that are common to both advanced planning and spontaneous decision making" (p. 131). For Simon, an HLT consists of three parts: "the learning goal, the learning activities, and the hypothetical learning process—a prediction of how students' thinking and understanding will evolve in the context of the learning activities" (p. 132).

Within the last decade the construct of a learning trajectory has received a great of attention because of the use of the idea in the Common Core State Standards for Mathematics (NGA & CCCSSO, 2010) and because of the popularity of learning progressions, a similar construct in the field of science education (e.g., Corcoran, Mosher, & Rogat, 2009). Recent research on learning trajectories within mathematics education (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Clements & Sarama, 2004, 2009; Confrey, Mahoney, Nguyen, & Rupp, 2009, 2014) is consistent with Simon's (1995) formulation to some degree. For example, Blanton and colleagues follow Clements and Sarama (2004) in characterizing a learning trajectory as involving "(a) learning goals, (b) instructional activities or an instructional sequence, and (c) a developmental progression that specifies increasingly sophisticated levels of thinking in which students might engage" (p. 514). Parts (a) and (b) are clearly similar to Simon's definition, and they are also contained within the definition of Confrey and colleagues.

However, this recent research has changed from Simon's (1995) initial formulation to focus on numerous levels of thinking through which students may pass, with less attention to "how students' thinking and understanding will evolve in the context of the learning activities" (p. 132). For example, Blanton and colleagues (Blanton et al., 2015) specify eight levels of thinking that ten first graders demonstrated during a set of instructional activities aimed toward provoking the students' development of functional relationships; Confrey and colleagues (Confrey et al., 2014) specify 16 proficiency levels in their equipartitioning learning trajectory; and Clements and Sarama (2009) describe 21 levels of thinking in their learning trajectory for counting (pp. 30-41). Although most certainly these trajectories involve the researchers in considering transitions between levels, in some cases the attention is more squarely on the levels themselves, and overall the scope seems large in comparison with Simon's use of the term. So, this research represents one way in which the idea of a learning trajectory has been elaborated in the field.

In contrast, Steffe's (2004) three-part definition of a learning trajectory is a model of students' "initial concepts and operations, an account of observable changes in those concepts and operations as a result of … interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes" (p. 131). In our view, this definition elaborates the idea of a learning trajectory differently from what we have described above, preserving Simon's (1995) scope but articulating the components and their relationships somewhat differently. That is, Steffe's first component of models is useful for specifying learning goals or a hypothetical learning process, Simon's first and third components. Steffe's second component, an account of changes that a researcher experiences in the models during mathematical interactions, is a basis for developing and refining learning activities, Simon's second component, and they are part of the learning process, Simon's third component. Steffe's third component, an account of the mathematical interactions that were involved in the changes, is also part of Simon's second and third components, because the mathematical interactions are part of learning activities as well as the learning process.

Furthermore, in our view, Steffe's (2004) three components for a single *epistemic* student could be seen as *one* piece of a "developmental progression of levels of thinking" (Clements & Sarama, 2004, p. 83), the third component in Clement and Sarama's definition (cf. Blanton et al., 2015). An epistemic student is not a particular student, but it is any but no particular student with certain cognitive characteristics deemed to describe a significant stage (von Glasersfeld & Kelley, 1981) in students' mental lives.[3] So, an epistemic student is a model of ways of thinking that are representative of students with particular cognitive characteristics. In our view, in Steffe's perspective on learning trajectories one has to develop his three components for different epistemic students (e.g., Hackenberg, 2014). Then, taken together, they could perhaps be seen as the third component of a learning trajectory identified by these other researchers.

In our research we have followed Steffe's (2004) definition of a learning trajectory because we are deeply engaged in constructing models of students' thinking using operations and schemes (e.g., Hackenberg, 2010b). Consistent with Simon's usage, we view a trajectory to be hypothetical when it is created ahead of interaction with students, and usually such hypothetical trajectories are more powerful if created based on prior interactions with students (Steffe & Thompson, 2000).

Method

Preparation

To prepare for the first experiment, the first author created three HLTs for each of three different epistemic students that she anticipated participating in the experiment. For example, for MC3 students she created an HLT for distributive reasoning with fractions and quantitative unknowns, an HLT for the construction of fractions as multipliers of quantitative unknowns, and an HLT for the construction of reciprocal reasoning with quantitative unknowns. All HLTs were informed by prior research (e.g., Hackenberg, 2010b, 2013, 2014; Hackenberg & Lee, 2015; Steffe & Olive, 2010; Ulrich, 2012).

Furthermore, the HLTs across epistemic students were connected. For example, for the MC2 students the first author created an HLT for reciprocal reasoning with unit fractions and quantitative unknowns, which was related to the HLT for reciprocal reasoning with quantitative unknowns for the MC3 students. For MC2 and MC1 students she created HLTs that were different from each other but both addressed additive inverse reasoning with quantitative unknowns.

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[3] An epistemic subject is "that which is common to all subjects at the same level of development, whose cognitive structures derive from the most general mechanisms of the co-ordination of actions" (Beth & Piaget, 1966, p. 308).

Participants

To launch the first experiment, we observed in mathematics classrooms at a local middle school and then implemented a two-part selection process with 21 seventh and eighth grade students: participation in a 30-minute interview and completion of a 12-item worksheet. The interview questions and worksheet were designed to assess students' multiplicative concepts and fractional knowledge. During the selection interview we also aimed to assess the student's openness to articulating her or his thinking and interest in participating in the design experiment. Following the interviews we consulted with classroom teachers about issues such as students' school attendance. Based on this process we invited nine students to participate: three MC3 students and six MC2 students.

Data Collection

Each of the 18 episodes ran after school on Tuesdays and Thursdays, lasted one hour, and was video-recorded with one stationary and two roaming cameras. Videos were mixed into a single file for analysis. During episodes students often worked in groups of two or three using a software program called JavaBars (Biddlecomb & Olive, 2000). Screenflow software was used to capture student computer work and conversation.

The first author served as the teacher for the episodes. Other team members operated roaming cameras, observed, took notes, and interacted with students. Between episodes the team processed data; kept an Episode Index consisting of summaries, observations, and conjectures; watched video and took notes; and discussed conjectures to prepare for the next episode. The teacher kept a reflective research journal. Following the experiment each student participated in a 45-minute, video-recorded interview to assess the student's current understanding of topics from the experiment and the student's experience of the class.

Data Analysis

For this paper we have developed second-order models of student participants, and we have also coded data twice. The two modes of analysis have been mutually informative.

Second-order models are constellations of constructs to describe and account for another person's ways of thinking (Steffe et al., 1983). To make these models we repeatedly viewed interview, small group, and whole classroom video files; transcribed major portions of video; examined student work; and took detailed analytic notes (Cobb & Gravemeijer, 2008). Because transcripts alone are limited in giving a feel for the classroom events (Zack & Graves, 2001), we watched video in conjunction with reviewing transcripts and wrote both data summaries and analytic notes on transcripts. We also wrote memos (Corbin & Strauss, 2008) consisting of interpretations of and conjectures about students' mathematical thinking and about interactions in which that thinking was evident. We traced back through the data to identify possible resources for constraints and changes in students' thinking (Cobb, et al., 2003). For example, when we noticed that a student iterated a segment but did not partition when drawing a picture, we looked back to how the student operated in the selection interview and other prior episodes to search for consistencies and possible changes in the student's activity. From the notes and memos we wrote portraits of the students that represent the second-order models.

At the same time, we performed an initial coding of transcripts from episodes 9-11 following grounded theory methods of open coding (Corbin & Strauss, 2008; Saldana, 2013). We met regularly to discuss and revise our codes, generating 106 codes. Since this number was cumbersome, we collapsed codes thematically to capture key features of our analysis, yielding 24 codes.

Then we performed a second coding on episodes 9-12 using ATLAS.ti. In this second iteration we united the coding for DI and the part of our second-order models focused on students' representations of multiplicative relationships among quantitative unknowns. In this process we expanded the number of codes from 24 to 37. In this second iteration of coding we also coded all relevant planning and reflection documents for the experiment, including the HLTs (Simon, 1995) developed prior to the experiment, the teacher's plans, conjectures in the Episode Index, and the teacher's research journal. At least two researchers coded each document, and we met regularly to discuss and revise our codes. We also wrote data summaries, memos, and conjectures about the development of DI in the four episodes. Out of this second coding we generated themes that led to the findings reported here.

Findings

Our analysis revealed five pedagogical activities that facilitated DI: (1) using HLTs in lesson planning; (2) providing students with choices; (3) monitoring actively during group work; (4) attending to small group functioning; and (5) conducting whole classroom discussions across different thinkers. The latter three pedagogical activities also, at times, impeded DI. In this section we elaborate on each of these pedagogical activities, giving examples from the data that illuminate it.

Using HLTs in Lesson Planning

At the start of the experiment the teacher (the first author) was experienced in developing second-order models relevant to the topics in the experiment. These second-order models were a foundation for creating HLTs as mentioned in the Methods section, and in turn, the HLTs guided both lesson planning and spontaneous decision making during episodes.

For example, the HLT for reciprocal reasoning with quantitative unknowns for the MC3 students, and the HLT for reciprocal reasoning with unit fractions and quantitative unknowns for the MC2 students, were the basis for planning learning activities during episodes 9-12. The teacher conjectured that MC3 students would construct reciprocal reasoning with quantitative unknowns by making accommodations in their iterative fraction schemes (Hackenberg, 2014).[4] In contrast, the teacher conjectured that MC2 students would construct reciprocal reasoning with only unit fractions because these students have yet to construct iterative fraction schemes (Hackenberg, 2007, 2014; Steffe & Olive, 2010).

The conjecture about the MC3 students proved to be very close to the actual learning trajectory for MC3 students across the three experiments (Hackenberg & Sevis, 2015). In contrast, the conjecture about the MC2 students required some adjustment; many MC2 students constructed what we call inverse reasoning with quantitative unknowns, but not reciprocal reasoning, even with unit fractions. Nevertheless, the HLTs provided an important guide in lesson planning and for being responsive during episodes that allowed the teacher to tailor instruction to students' thinking.

For example, the teacher conjectured that both MC2 and MC3 students would be able to work on representing two unknown heights where one height was a known whole number multiple of the other. However, based on a prior interview study she conjectured MC2 students would need more support in this process (Hackenberg, 2014; Hackenberg & Lee, 2015). In the prior study some MC2 students found drawing a picture of the unknown heights challenging, and most did not write multiplicative equations. Since that study was not a design experiment to provoke and understand learning, in the current study the teacher aimed to design problems and questioning support that would allow MC2 students to eliminate these difficulties. The subsequent sub-sections demonstrate this design.

Similarly, based on this same interview study (Hackenberg & Lee, 2015), the teacher conjectured that MC3 students were poised to represent in pictures and equations two unknown heights with a fractional relationship between them and to construct a quantitative meaning for a reciprocal (Hackenberg, 2014).

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[4] For students who construct iterative fraction schemes all fractions are multiples of unit fractions. In particular, an improper fraction like 7/5 is 7 x 1/5, as well as 1 whole and 2/5 (Hackenberg, 2007; Steffe & Olive, 2010).

However, she did not know how common this construction would be among MC3 students, nor how durable. So, she aimed to design problems and questioning support that would allow MC3 students to construct and generalize their reciprocal reasoning with quantitative unknowns.

In addition, because the HLTs for the two groups of students were related, they provided opportunities for the teacher to conduct and be responsive during whole classroom discussions across different ways of thinking. Although we discuss that pedagogical activity in a separate sub-section, we mention it here to show that the HLTs facilitated both goals of tailoring instruction to students' ways of thinking while also helping to build classroom community. Thus, using the HLTs facilitated DI.

Providing Students with Choices

The initial activity for some topics in the experiment involved providing students with choices of problems. The rationale for providing choices involves at least three issues. First, students may feel some empowerment or motivation if they can choose what to work on (Heacox, 2002). Second, students may choose a problem that is a good challenge to them, if they know that the teacher is purposefully writing problems that are supposed to be accessible to different thinkers in the class. Students might not always make the choice the teacher would make for them, but, third, student choices themselves are a useful piece of formative assessment for the teacher.

As an example, in episode 9 the teacher used *parallel tasks* (Small & Lin, 2010), which are a set of 2-3 problems designed to target different levels of thinking yet address similar mathematical ideas. In episode 9 the two problems were similar in structure and focus but differed based on what the teacher conjectured to be accessible to students with different multiplicative concepts (Figure 1), as discussed previously.

> **I. Corn Stalk Tomato Plant Heights Problem.** There is a tomato plant and stalk of corn growing in the garden, each of unknown height. The height of the stalk of corn stalk is 5 times the height of the tomato plant.[5]
> a) Draw a picture of this situation and describe what your picture represents.
> b) Write an equation for this situation that relates the two heights. Explain what your equation means in terms of your picture.
> c) Can you write another, different equation that relates the two heights? Explain what your equation means in terms of your picture.
> d) If you wrote an equation using division, can you write it with multiplication? Explain what your new equation means in terms of your picture.
> e) Let's say that the stalk of corn's height is 150 cm. How tall is the tomato plant?
> Use this example to check all of your equations.
> If an equation does not work, see if you can change it so that it does.
> Explain any changes that you make.
>
> **II. Tree Heights Problem.** Next to the school are two trees, each of unknown height. The crabapple tree is 3/5 the height of the maple tree.
> a) Draw a picture of the situation and describe what your picture represents.
> b) Write an equation for this situation that relates the two heights. Explain your equation in terms of your picture.
> c) Can you write another, different equation that relates the two heights? Explain this equation in terms of your picture.
> d) If you wrote an equation using division, can you write it with multiplication? Explain what your new equation means in terms of your picture.

[5] In subsequent experiments we specified that the heights were measured in the same units, e.g., centimeters.

e) Let's say that the maple tree's height is 20 feet. How tall is the crabapple tree?
Use this example to check all of your equations.
If an equation does not work, see if you can change it so that it does.
Explain any changes that you make.

Figure 1. Parallel Tasks used in episode 9.

The **Corn Stalk Tomato Plant Heights Problem** (**CS TP Hts Problem**) was designed to be accessible across both MC2 and MC3 students because of the whole number relationship between the two unknown heights, and the **Tree Heights Problem** was designed to be accessible to MC3 students because of the fractional relationship between the two unknown heights.

Students' work on these problems informed how we organized small groups in subsequent episodes. In episode 9, all students chose the **CS TP Hts Problem**. That was a legitimate choice for all, even though the teacher believed that MC3 students would be better challenged by the **Tree Heights Problem**.

However, despite choosing the same problem, the students worked on it in quite different ways. We observed three types of responses to parts (a) through (c), which is as far as students got on the problem in episode 9. Three MC2 students drew pictures in which they did not, from our perspective, show a 5 times relationship between the two heights (Figure 2). We'll call this group of students group I.

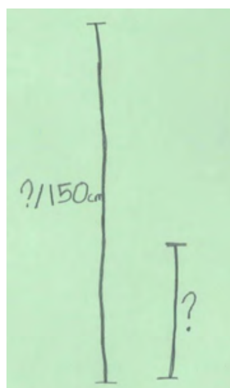

Figure 2. An example of an initial drawing by an MC2 student in group I for the **CS TP Hts Problem**.

Students in group I also did not necessarily write equations that reflected a 5 times relationship. For example, one of these students wrote "5 + x = h" and "h – x = 5", where h and x were not defined.

The three MC3 students clearly showed a 5 times relationship between the two heights in their pictures and equations (Figure 3). We'll call this group of students group II.

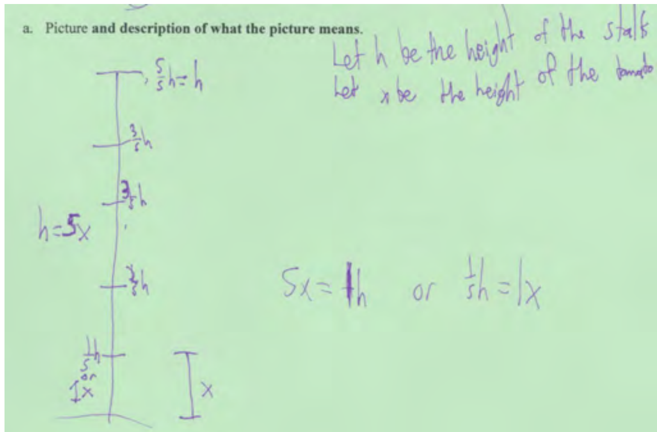

Figure 3. An example of an initial drawing by an MC3 student in group II for the **CS TP Hts Problem**.

Two MC2 students showed a 5 times relationship between the two heights in their pictures but we had some questions about their pictures and equations. For example, one student drew a picture that initially did not indicate a 5 times relationship but that looked fairly accurate (Figure 4, left). When that student was asked to show the relationship, he partitioned his longer height into 6 parts, not 5 (Figure 4, right). We'll call this group of students group III.[6]

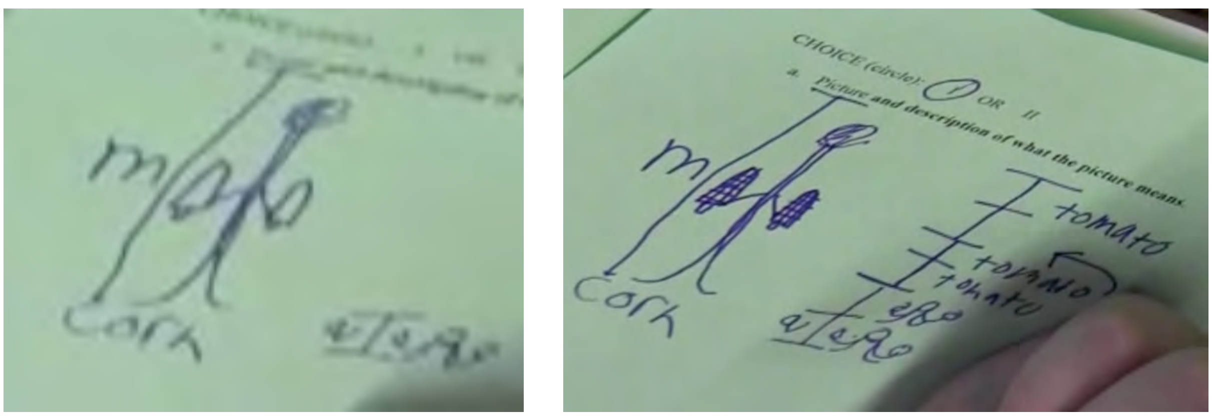

Figure 4. An example of an initial drawing by an MC2 student in group III for the **CS TP Hts Problem**.

In episode 10 we grouped together students according to their drawings and initial equations, that is, into groups I, II, and III, which meant that students with similar multiplicative concepts worked together. Based on the HLTs mentioned in the prior section, as well as on our observations of their work during the experiment, we designed questioning support about the **CS TP Hts Problem** for groups I and III, and we designed subsequent problems tailored to our interpretations of the thinking in all three groups. This kind of tailoring of instruction is a form of tiering instruction in which a teacher makes choices about different problems and activities that are likely to support the work of particular groups or individual students in a classroom (Tomlinson, 2005). Thus, student choices and student work allowed us to tailor instruction to students' learning needs, facilitating differentiation.

Monitoring Actively During Group Work

During episodes the teacher wove in and out of different groups where she asked questions to understand and support student thinking and to foster student-to-student communication. As we mentioned in the review of literature, this active, in-the-moment monitoring is characteristic of supporting group work and

[6] One MC2 student was absent during episode 9.

student learning in classrooms where responsive teaching occurs (e.g., Jacobs & Empson, 2015). However, the challenge is heightened with DI because groups are often working on different problems. So, the teacher needs to "tune in" to each group, act responsively in the moment, and frequently assess whether the planned tasks for the group should be the actual tasks.

This pedagogical activity often facilitated differentiation by allowing us to tailor instruction to students' learning needs. For example, in episode 10 when the students worked in groups I, II, and III on the **CS TP Hts Problem**, the teacher began with group I because those students had not shown a 5 times relationship in their pictures (from the research team's perspective, see Figure 2). Unfortunately one student, Lucy, was absent, so just two students worked together, Paige and Andrea.

The teacher told Paige and Andrea that they both had a good start on pictures "that kind of shows the relationship but might be able to show it more." She asked Andrea about how tall she thought her tomato plant was in her drawing compared to the corn stalk (Figure 2). When Andrea said, "five times smaller," the teacher asked if the picture showed that, and Andrea said, "no." The teacher also asked Paige to clarify her initial picture (Figure 5, left) by showing the relationship "more precisely." Following this interaction, both students changed their drawings to demonstrate the relationship, although they did so in different ways. In the following excerpt we show only Paige; elsewhere we provide analysis of both students' mathematical thinking in revising drawings and creating equations (Hackenberg, et al., in revision).

Data Excerpt 1: Paige demonstrates a revised drawing in episode 10 on 10/10.
P [to the teacher]: I made a different picture.
T: Oh, okay let's see.
P: These little marks right here mean like, um, I took my finger and went like this [pinches space between fingers, moves the space upwards along the larger plant height in her drawing, Figure 5, middle] and see how many times it fits into...
T: Oh, cool! I see, so this is what height? Which height is that, tomato plant or corn stalk? [P writes labels on drawings, Figure 5, lower right.] Awesome. Yeah, I can really see the relationship there in the picture Paige.

Figure 5. Paige's original drawing (left), process (middle) and revised drawing for the **CS TP Hts Problem** (right).

Paige iterated a small height five times to produce a larger height, making a completely new corn stalk height compared to her original drawing (Figure 5, left). In contrast, Andrea shortened her tomato plant height to (see segment marked "*a*" in Figure 6, lower right) and iterated it to try to fit five of those segments into her original corn stalk height. She wrote “more or less” next to her revised drawing, which indicates to us that she was aware of not quite accomplishing equal parts. So, although the students did not follow the same route to revising their drawings, they both made revisions that allowed them to show a 5 times relationship more precisely. In addition, as analyzed elsewhere, they used their drawings with further questioning support to develop multiplicative equations (Hackenberg, et al., in revision). Thus, the teacher appeared to tailor instruction well with Paige and Andrea.

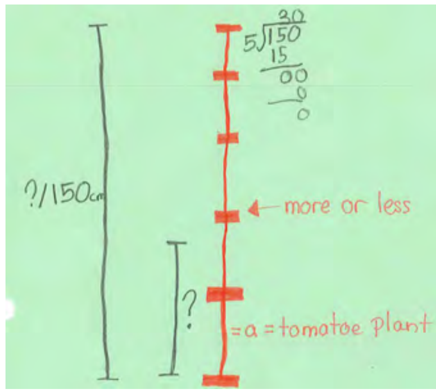

Figure 6. Andrea’s revised drawing for the **CS TP Hts Problem**.

With respect to the findings from Dyer and Sherin (2015) on the responsive teaching of secondary mathematics teachers, the teacher practiced all three types of instructional reasoning with Paige and Andrea. Here we highlight the third move, engaging in tests about student thinking, since as Dyer and Sherin state, “developing tests is perhaps one of the types of instructional reasoning most obviously related to the needs of responsive teaching because it enables teachers to gain more accurate and nuanced understandings of students’ reasoning and thinking” (p. 12 of pdf). Prior to interacting with Paige and Andrea in episode 10, the teacher had conjectured that both students would be able to iterate a small segment to produce a larger segment and so would be able to use that to refine their pictures of the **CS TP Hts Problem**. She tested this conjecture in the interaction, although not by directly asking the students to iterate. Instead, she asked them if they could revise their drawings to show the relationship “more precisely,” and they used iteration, thereby confirming her conjecture. In the experiment, this kind of testing of conjectures occurred frequently in both planned and spontaneous ways.

However, the pedagogical activity of monitoring actively could also impede differentiation when the teacher’s comments were not as responsive to students. For example, in episode 11 Lucy returned to class and worked with Paige and Andrea on two problems similar to the **CS TP Hts Problem**. To show a 3 times relationship between two unknown heights Lucy drew a taller segment partitioned into two parts and a shorter segment the size of one of those parts (Figure 7, left), and to show a relationship of ¼ between two unknown heights Lucy drew a taller segment partitioned into 3 parts and a shorter segment about the size of one of those parts (Figure 7, right).

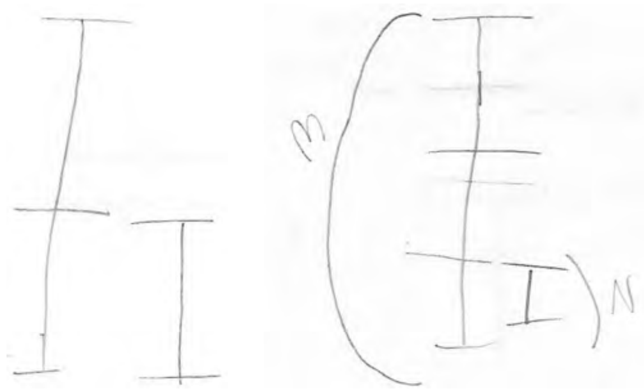

Figure 7. Lucy's pictures for a 3 times relationship (left) and a 1/4 relationship (right).

During episode 11 the teacher missed asking Lucy about her pictures for these problems. Although in episode 12 the teacher did ask Lucy about them, the omission in episode 11 meant that Lucy did not have the opportunity to rethink and activate ideas about the problems as Paige and Andrea had done. This kind of oversight is not uncommon even when all students are working on the same problems in a classroom, because it is easy to miss features of students' activity when balancing the many demands of teaching (Boaler & Humphreys, 2005). However, we view this oversight as being potentially more prevalent during DI because of the added diversity of activities and student work.

Attending to Small Group Functioning

The small groups varied in how they talked about their ideas and in how they got along. Sometimes their conversations supported them in thinking and working, which facilitated differentiation. For example, as Paige and Andrea continued on with their work in episode 10, they discussed their equations. In this process Paige seemed to come to a new understanding of how and why to write a multiplicative equation for the situation (Hackenberg, et al., 2015).

However, when groups had difficulty working together it could impede differentiation: Students did not seem to support each other in thinking or develop a sense of connection to each other. As an example we present three students, Gabriel, Martin, and Stephanie, who seemed to experience difficulties when working together—to the point that they were aware of the difficulties. One characteristic of their interaction was that they did not cohere together for long periods of time—they tended to work individually and come together for briefer time segments. Another characteristic of their interaction was territorial: They would verbally "spar" over who was going to use the computer, who had control of the mouse, and who made what contributions.

To give a flavor of what the interaction was like, we show an excerpt of portions of episode 10 when the students were working together to draw a picture on JavaBars for the **CS TP Hts Problem**. At the start of this excerpt, Stephanie was using the trackpad to draw a picture in JavaBars. Martin was vigorously asserting that he did not want Stephanie to color bars and parts on JavaBars; his comments stemmed from a prior episode in which Stephanie had spent considerable time coloring.

> **Data Excerpt 2:** Gabriel, Martin, and Stephanie's group work during episode 10 on 10/10.
> M: Gabriel, if she goes near the color [on JavaBars], just tell me, I'm taking them [the colors] out. [Martin reaches for the mouse; Stephanie is using the trackpad, and Gabriel is as well.]
> G: Okay, I understand Martin, thank you. Can you stop taking the mouse right now? I'm trying to hit okay. Thank you. [Gabriel growls at Martin who is using the mouse.]
> M [sees Gabriel's expression of frustration]: Sorry, okay. [Stephanie laughs at their interactions.]

G: Okay, so. [Martin uses the mouse again, dragging it along the table.] Oh my god Martin, will you quit it? I am serious. I'm going to disconnect your mouse. You can't handle this responsibility. Okay, just, okay. Stop! Stop! Just stop. You—
M: I promise. I do promise.
G [in a funny voice, indicating that he is partly joking]: No, you do not promise because you never do promise. You never do keep your promises.

[About 5.5 minutes pass, and the teacher comes over to ask how things are going.]

Tchr: How are you guys doing?
M [to the teacher]: It's good but she keeps wanting to fill [color] it, and we don't want to fill it because [overlapping with S below].
S: No, I helped. I cut that into pieces, and I wrote all this—
Tchr: Oh my, the same issue.
G: You did not cut this into pieces; I cut it into pieces!
S: I wrote all of this.
G: Nope, he wrote it.
M: I wrote it.
S: I wrote these two things.

Another feature of their interaction was that the boys excluded Stephanie. Stephanie was not always easy to include because she often focused on having her own work recognized as "smart" and the best. In addition, the boys had formed a bond—they seemed to like each other, despite the sparring. Sometimes Stephanie addressed her exclusion openly; for example, she stated, "I can't see because you guys aren't including me." Shortly after that Stephanie stood up and moved around to Martin's side of the table so that she could see the computer screen. Martin asked her to go back to her seat and adjusted the computer to give her a better view. However, Stephanie still seemed to feel excluded; Martin and Gabriel continued to spar in an edgy, playful way; and Stephanie called their group a "dictatorship" where Martin was dictator. When the teacher came over mid-episode Stephanie remarked, "We're having team work issues." The teacher tried to facilitate discussion and focus—and in fact, near the end of the episode both Martin and Gabriel engaged in the initial construction of reciprocal reasoning with quantitative unknowns, an act of learning that they seemed to appreciate (Hackenberg & Sevis, 2015). However, the difficulties in interaction persisted and seemed to interfere with Stephanie's learning and the development of connection. As Martin and Gabriel completed a problem near the end of the episode Stephanie said, "While you guys were doing all that mathematical stuff, I was drawing a rainbow and a happy sun."

As a result, in the later two experiments we found a need to help students to develop group autonomy by defining roles for group interaction (e.g., Boaler & Staples, 2008; Featherstone et al., 2011) and more explicitly discussing what behaviors could support and limit group work. For example, we instituted the idea of calling a teacher over for group questions only, which meant that students needed to ask groupmates first, with the intention that this might support groups in exploring ideas together. We also engaged in periodic check-ins called "Overheard Last Time," in which we looked together at some anonymous statements from groups during the last session and examined them for whether they supported or limited group work and why. We are in the process of analyzing the possible influences of these changes on the development of small group autonomy. Identifying the importance of supporting small group autonomy is a relatively new finding, in that literature has emphasized the need for individual autonomy with DI (e.g., Laud, 2011), but not necessarily for group autonomy.

Conducting Whole Classroom Discussions across Different Thinkers
The teacher conducted frequent whole class discussions across different ways of thinking and different problems. She was explicit about this: She tried to establish a norm that all students in class had

something to contribute to the discussions. For example, in episode 10 she spoke about how everyone in the classroom—the teacher, research team members, and students—had something to learn from others because everyone thought differently. These discussions allows students to share ideas that might have remained hidden in a non-differentiated classroom (Hackenberg, et al., 2015) and opened opportunities for students to organize their ideas and become more aware of them (Hackenberg & Sevis, 2015), both of which facilitated tailoring instruction and the development of a cohesive classroom community.

For example, in episode 11 the teacher held a whole classroom discussion about the work on the **CS TP Hts Problem** because most students had completed that problem in episode 10. To the teacher's and research team's surprise, MC2 student Tim insisted that the relationship between the two heights in the problem was "approximate" (Hackenberg, et al., 2015). The teacher pursued a discussion about this idea, in which 7 out of the 8 students participated actively (one student was absent). The students did not all agree with Tim, but his idea seemed to animate them to contribute. In speaking about it, students considered and expressed their own views in relation to Tim's, and some appeared to relate to his idea. For example, another MC2 student Lucy began her comment by stating, "Well, I understand his, what he's saying." Thus in holding the discussion, students had opportunities to articulate and clarify their own ideas, a feature of tailoring instruction to students' thinking. In addition, because students were responding to Tim's idea in a variety of ways, we view this whole classroom discussion as contributing to the development of a cohesive classroom community.

That said, such discussions are delicate. In episode 11, MC3 student Gabriel and MC2 student Tim were the most vocal in expressing opposing views (Hackenberg, et al., 2015). Gabriel seemed to think that Tim's idea was wrong; although he did not say so directly, he argued forcefully that a relationship between two unknowns could be exact and known. Tim seemed to interpret his own view as being in agreement with Gabriel, at least part of the time. So, although the conversation seemed stimulating for the students, it's not clear that these two students really understood the other person's perspective well. Navigating disagreement is not easy, particularly when the different viewpoints may not be easily reconciled. In our analysis, Gabriel did not understand why Tim would think the way that he did—i.e., it seemed obvious to Gabriel that a relationship between two unknowns can be exact and known. And, our current interpretation is that Tim's view was one outcome of operating with the second multiplicative concept, which meant that it would not necessarily be easily changed. So, under the surface of the discussion in episode 11 was the potential of two students talking past each other, which could lead to an impasse—a sense of distance rather than connection. Nevertheless, in episode 11 the classroom community did not seem particularly threatened by the discussion, perhaps because it was the first discussion of this type for this unit and students seemed interested and engaged.

However, in episode 12 the teacher led another whole classroom discussion based on student work from the prior episode on a problem in which one unknown height was ¼ of another unknown height. This work was posted to the board (Figure 8). The teacher began the discussion by asking, "can you write the equation $q/4 = c$ using multiplication?" In this equation, q represented the larger height and c represented the smaller height. MC2 student Connor stated an equation using his own letters, "$4C = B$". The teacher noted that he was doing something to the smaller height to produce the larger height, and what she was wondering was whether it was possible to write the equation $q/4 = c$ by multiplying the larger height by something to produce the smaller height. Gabriel immediately raised his hand. The teacher asked him to wait in order to let other students think about the question.

Figure 8. Student work posted to the board in episode 12.

Data Excerpt 3: MC2 students try to make sense of the question in episode 12 on 10/17.
Tchr: So that's the question, can you multiply the larger height by a number and get the smaller height. [Gabriel is waving his hand; Connor puts up his finger with an audible in-breath.]
C [putting down his finger]: Nooo [drawn-out, emphatically].
Tchr: So I know Gabriel has something he wants to say but I want to get everyone's ideas.
T: Unless you do two steps.
Tchr: Because some people have thought about this for longer than others. So what do you think Tim, unless you do two steps?
T: Well you could do, uh, [shrugs, pauses]. Hold on [short pause]. Multiply the larger number?
Tchr: Yeah, multiply q, our larger height by something to produce the smaller height [Connor puts up his hand]. Is it possible [Gabriel raises his hand] to do that and if so, how do we do it? [Martin raises his hand too.]
T: Well, it's possible, but I don't know what the…
Tchr: You don't know what would work. What do you think Connor?
C: Maybe you could multiply it like, by itself and then divide it by something [Martin's hand is up higher now, and Gabriel's is down.]

The conversation continued. When asked again whether it would be possible to write the equation by using only multiplication, Connor's response was "No!" Tim thought it was possible but was not sure what would work and then suggested using scientific notation. Lucy did not suggest ideas and nodded when the teacher asked if it was a strange question. So at this point, the question appeared to be puzzling for MC2 students, while MC3 students continued to raise their hands to respond.

In the next part of the discussion, Gabriel said they could write ¼ of q equals c, or ¼ x q = c, and Martin agreed. The teacher asked Lucy, Connor, and Tim whether that seemed sensible. Lucy said it did, "Because you would have to, um, multiply by the reciprocal which would be four, and then you multiply the reciprocal on other side and you get the same exact answer." So her reason seemed to focus on rules for creating equivalent equations. Connor said he had thought of something about fractions before but did not say it, but he was no more specific than that.

Then, the students discussed what multiplying by a fraction and dividing by a fraction might mean. Martin said, "multiplying by 1/4 and dividing by 4 is the same thing." The teacher asked how many agreed with Martin, and Tim, Connor, and Lucy all put their thumbs "in the middle" to indicate "sort of." A short while later the teacher and students were considering $q \div ¼ = c$, because Lucy had written this equation (with different letters) on her paper in episode 11. The teacher asked how they saw the equation in the picture. Lucy smiled and shrugged. Connor said, "if you were to divide the smaller height by ¼, it'd be like dividing it by 4, so then it'd equal the smaller height." The teacher asked whether he saw dividing by ¼ and 4 as the same and Connor said, "Yeah!" Martin immediately disagreed, stating that because dividing q by ¼ meant multiplying q by 4; Gabriel agreed with him.

We stop description at this point to give some analysis. The whole classroom discussion in episode 12 seemed like a bigger challenge to the development of community because it was another, perhaps more obvious surfacing of two viewpoints that could not be easily reconciled. The MC3 students seemed to know immediately how to turn a division equation into a multiplication equation by using fractions as multipliers (although it is likely their ideas about this were largely procedural; they did not have a way to justify beyond just knowing a rule). In contrast, the MC2 students seemed quite puzzled by the question and, despite Lucy's comment about creating equivalent equations, there was no evidence that they had strong reasons to agree with the MC3 students' ideas. As the conversation continued, it revealed that dividing by 4 and dividing by 1/4 was undifferentiated for the MC2 students.

Since the teacher could see that the viewpoints were not easily reconciled, she made two moves in the moment. First, she asked, "How do you see $q \div ¼ = c$ in the picture?" None of the students were readily able to see the division in the pictures they had drawn, so doing so remained a challenge for both MC2 and MC3 students. The teacher used this puzzlement as an opportunity to restate a goal in the class, which was to always try to see their equations in their pictures. Second, she said that they would continue to work on the issue of how to write division equations using multiplication, indicating the topic was something to work on. The first move was an attempt to bring students together by challenging them all together and returning to a class goal; the second move was an attempt to communicate that there was no expectation for everything to get resolved after one discussion. In addition, during planning the research team decided to use heterogeneous groups for the next few episodes that addressed a new topic, to avoid emphasizing a difference that might have made the MC2 students feel less capable. Although these in-the-moment and planned moves may have helped to maintain community for this particular class, we view them as examples of how tenuous and problematic it can be to conduct whole classroom discussions across different thinkers, and how doing so can threaten classroom community.

Emergent Theory of DI

In this section we reflect on how our findings inform our emergent theory (Cobb & Gravemeijer, 2008) of DI in mathematics for cognitively diverse middle school students. We aim for this theory to be applicable across research settings and classroom teaching. Given the commonalities in responsive teaching moves in classrooms and in research settings involving one-on-one interaction with students (Dyer & Sherin, 2015; Jacobs & Empson, 2015), we feel that this aim may be achievable. However, here we speak about the theory as derived from a research setting and comment on what we think is similar to and different from classroom environments. We note that currently we are focused on what we view as the heart of DI; our emergent theory does not yet encompass all of our findings about DI.

Our analysis and other activities of the IDR^{2}eAM project, such as a Teacher Study Group currently in progress, indicate to us that the heart of DI in mathematics for cognitively diverse middle school students is teachers getting to know students' thinking (Figure 9). To get to know students' thinking requires, at minimum, strategies for asking questions and posing problems that would allow students' thinking to become visible (Mason, 2008; Lampert, 1990; Steffe & Thompson, 2000). Such questions and problem-posing are a feature inherent to design experiments (Cobb & Gravemeijer, 2008).[7] They are not always a feature of typical middle school classroom instruction.

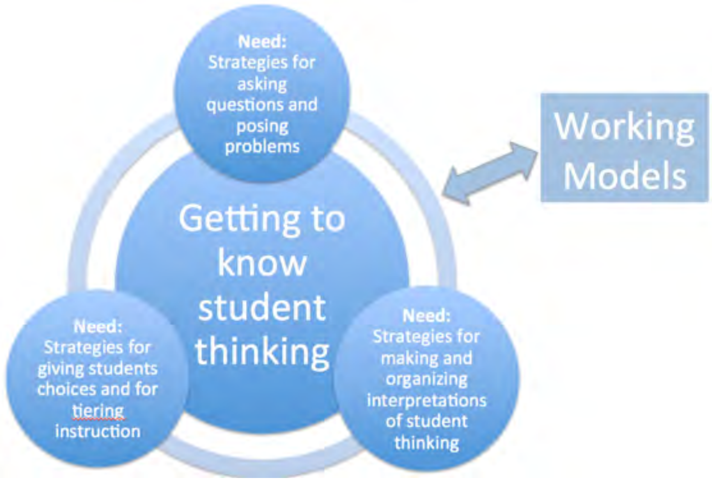

Figure 9. Emergent theory of DI in mathematics for cognitively diverse middle school students.

A second aspect of getting to know student thinking involves strategies for providing choices for students and for tiering instruction. As noted, choices students make are themselves a piece of formative assessment. The choices combined with the work that students do on their selected problems and activities are a basis for teacher choice of tiered problems for students. These strategies are not a built-in feature of design experiment methodology or typical middle school classroom instruction. In our view, using these strategies can facilitate tailoring instruction and support making student thinking visible.

A third aspect of getting to know student thinking requires strategies for making interpretations of student thinking and organizing those interpretations. Teachers who aim to differentiate must make interpretations of students' thinking in order to use that thinking as a basis for instructional decisions (both planned and in-the-moment), but not all interpretations have equal power or usefulness. So, teachers who aim to differentiate have to weigh alternative interpretations. Teachers also must organize these interpretations so that they can draw on them to inform their interactions with students.

For us, this process represents the creation of working models of students' mathematical thinking (Steffe & Thompson, 2000). As mentioned previously, we view working models as generative, dynamic tools for interaction that are usually less analytical than second-order models.[8] In our research we use schemes and operations that we attribute to students as basic components of our working models. However, we don't expect classroom teachers' working models to necessarily be composed similarly. A key question for us, currently, is about the nature and development of teachers' working models of students.

Toward this end we have found useful some case studies of expert, responsive elementary teachers in the literature on Cognitively Guided Instruction (CGI) (e.g., Fennema, Franke, Carpenter, & Carey, 1993; Jacobs & Empson, 2015; Steinberg, Empson, & Carpenter, 2004). For example, Kathy Statz (Steinberg, et al.) developed what we recognize as working models of students' mathematical thinking during her third year of teaching. Ms. Statz began her career as a teacher curious about her students' thinking and well-

[7] This does not mean achieving the goal of making student thinking visible is easy to accomplish!

[8] Steffe and Thompson (2000) use the term "living models" and define them as follows: "Operations and meanings we impute to students within the boundaries of their essential mistakes" (p. 277). Essential mistakes refer to ways of thinking that persist in students despite researchers' sincere efforts to help students to eliminate them.

versed in CGI, but in her first two years of teaching she did not use her students' thinking as a site for investigation or to develop whole classroom discussion. Over a relatively short period of time in her third year she went through four phases of change that resulted in her developing working models that allowed her to engage in responsive teaching (Jacobs & Empson, 2015) and use students' thinking to develop productive whole classroom discussions. We see these working models as the kind needed for DI.

Since Ms. Statz (Steinberg et al., 2004) was a teacher within the CGI program, we know that a component of her working models was her understanding of CGI frameworks, which includes problem structures, students' solution strategies, and what those strategies likely indicate about students' thinking (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). But that cannot be the only component, since she seemed to develop working models when she interacted with students intensively and in an investigative manner during her third year of teaching. We conjecture that her working models likely also included abstractions about particular ways of thinking she saw students' use repeatedly, ideas about pathways (or trajectories) that students often follow when working on particular big ideas (such as moving from counting by 1s to organizing numbers into composite units of 10s and 1s), and questions that might support students' progress along those pathways. Further, she likely had an eye out for students who seem different from students she has seen before. We wish to know more about teachers' working models at the secondary level in order to continue developing our emergent model of DI.

We emphasize that our emergent theory is truly emergent in that it does not yet address other important aspects of DI, such as management of formative assessment data (which is challenging as numbers of students increase), efforts to support individual and small group autonomy, and the large issue of how to conduct productive whole classroom discussions across different problems and different ways of thinking. This issue has begun to be addressed by Leatham and colleagues (Leatham et al., 2015) who have developed a method of locating productive opportunities in the flow of classroom experience for teachers to build on students' mathematical thinking. Although they are not focused on DI, their method could prove useful for navigating this terrain when the focus is on DI.

Concluding Thoughts

In this manuscript we addressed the research question: **How did pedagogical activities facilitate and impede differentiating mathematics instruction for middle school students in an after school design experiment?** We found that using HLTs in lesson planning, providing students with choices of problems, monitoring actively during group work, attending to small group functioning, and conducting whole classroom discussions across different thinkers all facilitated DI. We also found that the latter three pedagogical activities could impede DI. In particular, DI was impeded when the active monitoring was not responsive enough, when small groups did not develop autonomy, and when the whole classroom discussions demonstrated viewpoints that were not easily reconciled. As a result, in our subsequent two experiments we made some changes, such as more explicitly supporting small groups to develop autonomy. However, we suspect that similar issues will continue to arise because of the challenge of DI.

These findings have helped us begin to shape our emergent theory of DI in mathematics for cognitively diverse middle school students, which we are eager to continue to develop as we continue our analysis.

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